UNIT-1 : ORDINARY DIFFRENTIAL EQUATIONS-I (ODE-I)

Syllabus : Ordinary Differential Equations-I: First-order differential equations (Separable, Exact, Homogeneous, Linear), Linear differential Equations with constant coefficients. Homogeneous linear differential equations, Simultaneous linear differential equations.

1. Define ODE, Order and Degree of ODE.
2. Write the order and degree of the following diff. equations: (i) $\frac{d^2y}{dx^2}+3(\frac{dy}{dx})^3+6y=3x^3$ (Order=2, Degree=3)

(ii) $\left(\frac{d^3y}{dx^3}\right)^4+3(\frac{d^2y}{dx^2})^3+6\frac{dy}{dx}+y=3x$ (iii) $\frac{d^3y}{dx^3}+3(\frac{d^2y}{dx^2})^3+6\left(\frac{dy}{dx}\right)^2+y=3x$

Ans: (i) Order=0 , Degree=1(ii) Order=3 , Degree=4(iii) Order=3 , Degree=1

Formation of ODE:

3. **Form the ODE** $y=A\cos ax+B\sin ax$ Ans: $\frac{d^2y}{dx^2}+a^2y=0$
4. **Form the ODE** $y=e^{ax}(A\cos ax+B\sin ax)$ Ans: $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$
5. **Form the ODE** $y=A\cos x^2+B\sin x^2$ Ans: $x\frac{d^2y}{dx^2}-\frac{dy}{dx}+4x^3y=0$ **[June 2006]**

SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFF. EQUATIONS:
Variable separable method:

Separation of Variables method

This method is used when the equation is in the simplest first-order form of equation

e.g. $\frac{dy}{dx}=\frac{g(x)}{h(y)}$ **(Basic form)**

1. **Separate the variables *y* from *x*, i.e., by collecting on one side all terms involving *y* together with *dy*, while all terms involving *x* together with *dx* are put on the other side.**
2. **Integrate both sides.**
3. **If the solution can be defined explicitly, i.e., it can be solved for *y* as a function of *x*, then do it. If not, the solution can be defined implicitly, i.e., it cannot be solved for *y* as a function of *x*.**

6. Solve $y\,dx-xdy=0$ **Ans:** $x/y=C$
7. Solve $\frac{dy}{dx}+y=1$ **Ans:** $x=-\log(1-y)+C$
8. Solve $x\frac{dy}{dx}+\cot y=0$ **[May 2018]**
9. Solve $\frac{dy}{dx}=1+x+y+xy$ **Ans:** $\log(1+y)=x+x^2/2+C$
10. Solve $3e^x \tan y\,dx+(1-e^x)\sec^2 y\,dy=0$ **Ans:** $(e^x-1)^3=C\tan y$
11. Solve $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$ **Ans:** $e^y=e^x+x^3/3+C$ **[Jan. 2007]**
12. Solve $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0$ **Ans:** $\sin x\,(e^y+1)=C$ **[June. 2007]**

13. Solve $y\,dx + (1+x^2)\tan^{-1}x\,dy = 0$ **Ans:** $y\tan^{-1}x = C$

14. Solve $\frac{dy}{dx} + 2\frac{y}{x} = \sin x$ **[Nov. 2019]**

HOMOGENEOUS DIFF. EQUATIONS:

Homogeneous Ordinary Differential Equations

A homogeneous ordinary differential equation is an equation of the form *P(x,y)dx+Q(x,y)dy=0* where P and Q are homogeneous of the same order.

Put *y= v x* and *dy/dx = v+ x dv/dx* in the given equation , and use seperation of variavle

15. Solve $xdy - ydx = \sqrt{x^2+y^2}\,dx$ **Ans:** $x^2 = c\left[y + \sqrt{x^2+y^2}\right]$ **[June02, 05,14]**

16. Solve $y - x\frac{dy}{dx} = x + y\frac{dy}{dx} = 0$ [**or** $(x+y)dy+(x-y)dx=0,\ y=1$ *at* $x=1$] **Ans.** $c(x^2+y^2) = e^{-2\tan^{-1}y/x}$

17. Solve $x(x-y)dy + y^2dx = 0$ **Ans:** $y = ce^{y/x}$ **[Dec. 04]**

18. Solve $(e^{x/y}+1)dx + e^{x/y}(1-x/y)dy = 0$ **Ans.** $x + ye^{x/y} = c$ **[RGPV Dec.2003]**

LINEAR DIFFERENTIAL EQUATIONS (LEBNITZ'S DIFFERENTIAL EQUATIONS)

METHOD-III : Linear Equation

This method is used when the equation is in the form of $\frac{dy}{dx} + p(x)y = q(x)$ **(Basic form)**

Where *p(x)* and *q(x)* – continuous functions may or may not be constants.

Solution: Find Integral Factor, I.F.= $e^{\int pdx}$ **, Then Solution :** $y.(I.F.) = \int I.F.Q(x)dx + C$

OR

$$\frac{dx}{dy} + p(y)x = q(y) \qquad \textbf{(Alternative form)}$$

where *p(y)* and *q(y)* – continuous functions may or may not be constants.

Solution: Find Integral Factor , I.F.= $e^{\int pdy}$ **, Then Solution :** $x.(I.F.) = \int I.F.Q(y)dy + C$

19. Solve $(y-x)\frac{dy}{dx} = a^2$ **Ans.** $x = (y-a^2) + Ce^{-y/a^2}$ ***[RGPV Dec. 2011]***

20. Solve $xdy - ydx + 2x^3dx = 0$ **Ans.** $y = -x^3 + Cx$ ***[RGPV June. 2011]***

21. Solve, $(1+y^2)dx = (\tan^{-1}y - x)dy$ **Ans.** $x = ce^{-\tan^{-1}y} + (\tan^{-1}y - 1)$ **[June.03,Feb.05,10 June08,March10, june 17]**

22. Solve $\frac{dy}{dx} = 1 + x + y + xy$ ***[RGPV Dec. 2011, June 17]***

23. Solve, $\frac{dy}{dx} - \frac{2y}{x+1} = (x+1)^3$ ***Ans.*** $y = c(x+1)^2 + \frac{}{2}$ ***[RGPV Dec.2006]***

24. Solve $\sqrt{1-y^2}\,dx = (\sin^{-1}y - x)dy$ ***Ans.*** $x = c.e^{-\sin^{-1}y} + \sin^{-1}y - 1$ **[RGPV June2007]**

25. Solve $\frac{dy}{dx} = -\frac{x + y\cos x}{1+\sin x}$ **Ans.** $y(1+\sin x) = c - \frac{x^2}{2}$ **RGPV Dec.2003]**

26. Solve *cosx.dy =(sinx-y)dx* ***Ans. y(secx+tanx) =c + secx + tan x-x)*** **[RGPV June2004,Sept.2009]**

27. Solve $\frac{dy}{dx} + y\cot x = 2\cos x$ **Ans.** $y.\sin x = c - \frac{\cos 2x}{2}$ **[RGPV June2004]**

28. Solve the equation subject to the condition, $\frac{dy}{dx} + \frac{y}{x} = x^2$,y=1 when x=0 ***Ans.*** $y = \frac{x^3}{4}$ ***[Dec. 2007May 18]***

29. Solve $\frac{dy}{dx} + y\tan x = \sin x$ **Ans.** **[RGPV Nov.18]**

30. Solve $\frac{dy}{dx} + 2\frac{y}{x} = \sin x$ **Ans.** **[RGPV Nov.19]**

BERNOULLI'S DIFFRENTIAL EQUATIONS (REDUCIABLE TO LINEAR EQUATIONS)

Bernoulli's Equations: The equation $\frac{dy}{dx} + p(x)\, y = g(x)\, y^a$ (a is any real number) which is known as the *Bernoulli's Equation*, can be reduced to linear form by a suitable change of the dependent variables ($u(x) = y^{1-a}$.)

31. Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$ **Ans:** $\tan y = \frac{1}{2}(x^2-1) + Ce^{-x^2}$ ***[Dec. 2005]***

32. Solve $x\frac{dy}{dx} + y = y^2\log x$ **Ans:** $y(1+\log x+Cx)=1$

33. Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$ **Ans:** $(2-y^2) + Ce^{-y^2/2} = \frac{1}{x}$ **[June. 2005, April 2009]**

34. Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x.\sec y$ **Ans:** $\sin y = (1+x)e^x + C(1+x)$ **[June 2009]**

EXACT DIFFERENTIAL EQUATIONS:

M(x,y) dx + N(x,y) dy = 0 If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ **then equation is called exact.**

Solution : $\int_{y=constant} M dx + \int_{x=constant} N dy = C$ **(write common terms once)**

35. Solve $(1+4xy+2y^2)dx + (1+4xy+2x^2)dy = 0$ **Ans: (x+y)(1+2xy)=c** **[RGPV Feb.1995,99, June 17]**

36. Solve $(e^{x/y}+1)dx + e^{x/y}(1-\frac{x}{y})dy = 0$ **Ans.** $x + ye^{x/y} = C$ **[RGPV June 2015]**

37. Solve $(e^y+1)\cos x dx + e^y \sin x dy = 0$ **Ans.** $(e^y+1)\sin x = c$ **[RGPV Dec.2000, June 2014]**

38. Show that the equation $(5x^4 - 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$ is an exact equation. Find its solution. **[Nov. 2018]**

39. Solve $ye^x dx + (2y+e^x)dy = 0$ **Ans.** **[RGPV Nov. 2019]**

DIFFERENTIAL EQUATIONS REDUCIBLE TO EXACT FORM:

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then equation is not exact :Now we have to convert the given equation in exact form. using following methods:

Method-I : If Mx+Ny≠0 (a small term) , then take I.F.(Integral Factor) = $\frac{1}{Mx+Ny}$, and multiply the equation by I.F. to reduce in exact form.

Methid-II If ***$f_1(x,y)y\,dx + f_2(x,y)x\,dy=0$*** , If ***Mx-Ny***≠0 (a small term) , then take I.F.(Integral Factor) = $\frac{1}{Mx-Ny}$, and multiply the equation by I.F. to reduce in exact form.

Methid-III: When $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x)$ function of x only, then $I.F.=e^{\int f(x)dx}$, multiply this I.F. to equation and make exact.

Methid-IV: When $\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-f(y)$ function of y only, then $I.F.=e^{-\int f(y)dy}$, multiply this I.F. to equation and make exact.

Method-V: General Method: If the diff. equation is of the form $x^a y^b(mydx+nxdy)+x^r y^s(pydx+qxdy)dy=0$
Then take $x^h y^k$ as the integral factor and multiply this I.F. in the given equation.
Apply condition of Exactness and find the values of *h* and *k*. Put these values in equation and solve it.

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40. Solve $(y+x-5)dx-(y-x+1)dy=0$ **[June 16]**

41. Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)dy=0$ **(Rule -I)Ans.** $\frac{x}{y}+\log(\frac{y^3}{x^2})=c$ **[Dec. 2002 ,Feb.06,Dec.08]**

42. Solve $x^2ydx=(x^3+y^3)dy$ **(Rule-I) Ans.** $-\frac{x^3}{3y^3}+\log y=C$ **[Dec. 2002 ,Feb.2006]**

43. Solve $y\sin 2xdx-(1+y^2+\cos^2 x)dy=0$ **(Rule-II)** ***Ans: 3ycos2x+6y+2y³=c*** **[RGPV Feb.1996]**

44. Solve $y(1+xy)dx+x(1-xy)dy=0$ **(Rule-II) Ans.** $x = yce^{1/xy}$ **[RGPV Dec.2003]**

45. Solve $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$ **(Rule-II) Ans.** $2\log x-\log y=\frac{1}{xy}+C_1$ ***[RGPV Feb.2012]***

46. Solve $(y^2-x)dx+(2y)dy=0$, **(Rule-III) Ans.**

47. Solve $(x^2+y^2+2x)dx+(2y)dy=0$, **(Rule-III) Ans.** $(x^2+y^2)e^x=C$

48. Solve $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$, **(Rule-IV) Ans.** $x^3y^3+x^2=cy$ ***[RGPV Feb.2001, June 2013]***

49. Solve $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$, **(Rule-IV) Ans.** $3x^2y^4+6xy^2+2y^6=0$

50. Solve $(x+y-2)dx+(x-2y-3)dy=0$ **Ans.** $x^2+2xy-4x-2y^2-6y=2c$ **[RGPV Dec.1999]**

51. Solve $(a^2-2xy-y^2)dx-(x+y)^2dy=0$ ***Ans:*** $a^2x-xy(x+y)-\frac{y^3}{3}=c$ **[RGPV DEC.2000]**

52. Solve $(2y+6xy^2)dx+(3x+8x^2y)dy=0$, **Ans.** $x^2y^3+2x^3y^4=c$ **[RGPV Dec. 2004]**

1. $a^m \times a^n = a^{m+n}$ **(2)** $(a^m)^n = a^{mn}$ **(3)** $(ab)^m = a^m b^m$ (4)
2. If $ax^2+bx+c=0$ and $a \neq 0$, the roots of this equation is given by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

LINEAR DIFFERENTIAL EQUATION OF HIGHER ORDER WITH CONSTANT COEFFICIENT

SECOND-ORDER DIFFERENTIAL EQUATIONS $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$, where *p(x)*, *q(x)*, and *r(x)* are continuous functions.

If ***r(x) = 0*** for all *x*, then, the equation is said to be ***homogeneous*.**

If ***r(x) ≠ 0*** for all *x*, then, the equation is said to be ***nonhomogeneous*.**

Solving Second-Order Linear Homogeneous Differential Equations With Constant Coefficients (When *r(x)* = 0):

Two continuous functions *f* and *g* are said to be *linearly dependent* if one is a constant multiple of the other. If neither is a constant multiple of the other, then they are called *linearly independent*.

To Find Solution of ***homogeneous*** **differential equation:**

Find "*auxiliary quadratic equation*" or "*auxiliary equation*" by Replacing $\frac{d^2y}{dx^2}$ with m^2, $\frac{dy}{dx}$ with *m*, and *y* with 1

Case- I : If *auxiliary equation* has **real and distinct** roots m_1 and m_2 then

Complementary Function, C.F.= $y = y_c(x) = c_1e^{m_1x} + c_2e^{m_2x}$

Case-II : If *auxiliary equation* has real and equal root $m_1 = m_2 = m$ then

C.F. = $y = y_c(x) = c_1e^{mx} + c_2xe^{mx} = (c_1 + c_2x)e^{mx}$

Case –III : If *auxiliary equation* has complex roots $m = \alpha \pm \beta i$ (i.e. $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$)then

C.F. = $y = y_c(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

Solution of Second-Order (or Higher order) Linear Nonhomogeneous Differential Equations With Constant Coefficients (When $r(x) \neq 0$)

The **General solution** of $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$ is $\boxed{y(x) = y_c(x) + y_p(x) = C.F. + P.I.}$

To Find P.I.

1	When *r(x)*= e^{ax}	Put *D*= *a* , except when *f(a)* ≠0	$P.I. = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$
2	When *r(x)*= e^{ax} (Special Case When ***f(a)=0***)	Put *D=D+a* and Solve the equation for 1= x^0 or e^{0x}	$P.I. = \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(D+a)}\ 1$
3	When *r(x)= sinax or cosax*	Put $D^2=-a^2$ and Solve the equation for *D* by rationalization of the equation (same for *cosax*) Except $f(-a^2) \neq 0$	$P.I. = \frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$
4	When *r(x)= sinax or cosax* (Special Case)	If *f(- a²) = 0 then use*	$P.I. = \frac{1}{f(D^2+a^2)}\sin ax = \frac{x}{2a}\cos ax$ $P.I. = \frac{1}{f(D^2+a^2)}\cos ax = -\frac{x}{2a}\sin ax$
5	When *r(x)*= x^m	Expand Series $f(D)^{-1}$ using $(1-x)^{-1} = 1 + x + x^2 + x^3......$ $(1+x)^{-1} = 1 - x + x^2 - x^3......$	$P.I. = \frac{1}{f(D)}x^m = f(D)^{-1}x^m$

F or Product of Two Functions :			
6	When $r(x)= e^{ax} V$ (Where V is the function of x)	Put $D= D+a$ for e^{ax} and then use given formua (For solving V use formula from 1 to 5)	$P.I. = \frac{1}{f(D)} x.e^{ax} = e^{ax} \frac{1}{f(D+a)} V$
7	When $r(x)= x V$ (Where V is the function of x)	For solving V use formula from 1 to 5	$P.I. = \frac{1}{f(D)} x.V = x.\frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$

General Method : $\frac{1}{f(D)} Q = \frac{1}{(D-\alpha_1)} Q = A_1 e^{\alpha_1 x} \int e^{-\alpha_1 x} dx$

OR

$$\frac{1}{f(D)} Q = \frac{1}{(D-\alpha_1)(D-\alpha_2)(D-\alpha_3)......(D-\alpha_n)} = [\frac{A_1}{(D-\alpha_1)} + \frac{A_2}{(D-\alpha_2)} + ... + \frac{A_n}{(D-\alpha_n)}]Q$$
$$= A_1 e^{\alpha_1 x} \int e^{-\alpha_1 x} dx + ... + A_n e^{\alpha_n x} \int e^{-\alpha_n x} dx$$

53. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ **Ans:** **: [Nov.2019]**

54. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ **Ans: y=** $(c_1 + c_2 x)e^{-1x}$

55. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$ **Ans: y=** $e^{-1x}(c_1 \cos 2x + c_2 \sin 2x)$

56. Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}$ **[Nov. 18, May 18]**

57. Solve $(D^2+1)(D^2-1)y = e^{2x} + x^2$ **Ans.**: $c_1 \cos x + c_2 \sin x + c_3 e^x + c_4 e^{-x} + \frac{1}{15} e^{2x} - x^2$ [**Dec.2002]**

58. Solve $(D^2 - 7D + 6)y = e^{2x}$ **Ans.** $y = c_1 e^x + c_2 e^{6x} - \frac{1}{4} e^{2x}$

59. Solve $\{(D-1)^2 (D-3)^3\} y = e^{3x}$ **Ans:** $y = (C_1 + C_2 x)e^x + (C_3 + C_4 x + C_5 x^2)e^{3x} + \frac{x^3 e^{3x}}{24}$ **[Dec 2010]**

60. Solve $(D^3+1)y = (e^x+1)^2$ **Ans.** $y = c_1 e^{-x} + e^{x/2}(c_2 \cos\frac{\sqrt{3}}{2} x + c_3 \sin\frac{\sqrt{3}}{2} x) + \frac{1}{9} e^{2x} + e^x + 1$

61. Solve $(D^2 + 4D + 3)y = e^{-3x}$ **Ans.** $y = c_1 e^{-3x} + c_2 e^{-x} - \frac{x}{2} e^{-3x}$

62. Solve $(D^4 - 3D^2 - 4)y = 5\sin 2x$ **[Nov 2018]**

63. Solve $(D+2)(D-1)^3 y = e^x$ **[Nov. 2019]**

64. Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$ **Ans:** $y = C_1 e^x + (C_2 + C_3 x)e^{-x} - \frac{1}{25}(\cos 2x + 2\sin 2x)$

65. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$ **Ans:** $y = C_1 e^{-x} + C_2 e^{-2x} + 1 + \frac{1}{10}(3\sin 2x - \cos 2x)$ **[Dec.02,June07]**

66. Solve $\frac{d^2y}{dx^2}+4y=e^x+\sin 2x$ **Ans:** $y=C_1\cos 2x+C_2\sin 2x+\frac{e^x}{5}-\frac{x\cos 2x}{4}$ **[june2012, June 17]**

67. Solve $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=\cos x+e^x$

Ans: $y=C_1e^x+e^x(C_2\cos x+C_3\sin x)+xe^x+\frac{1}{10}(3\sin x+\cos x)$ []**Dec.2012 June 17]**

68. Solve $(D^2+4)y=x^2$ **Ans**. $y=c_1\cos 2x+c_2\sin 2x+\frac{x^2}{4}-\frac{1}{8}$

69. Solve $(D^3+3D^2+2D)y=x^2$ **Ans**. $y=c_1+c_2e^{-x}+c_3e^{-2x}+\frac{1}{12}x(2x^2-9x+21)$ [**June 06, 2015]**

70. Solve $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=\cos 2x+3e^x$ **[June 16]**

71. Solve $(D^2+4)y=x^2+\cos^2 x$ Ans. $y=c_1\cos 2x+c_2\sin 2x+\frac{x^2}{4}+\frac{x}{8}\sin 2x$

72. Solve $(D^3+4D^2+D)y=e^{2x}+x^2+x$ **Ans** $y=c_1+(c_2x+c_3)e^{-x}+\frac{e^{2x}}{18}+\frac{x^3}{3}-\frac{3x^2}{2}+4x$ [**June 04].**

73. Solve $(D^2-4D+4)y=3e^x+x^2+\sin 2x$ Ans. $y=(c_1+c_2x)e^{2x}+\frac{1}{4}(x^2-2x+\frac{3}{2})+3e^x+\frac{\cos 2x}{8}$ [**June2011]**

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74. Solve $(D^3-3D+2)y=540x^2e^{-x}$ **Ans**. $y=(c_1+c_2x)e^x+c_3e^{-2x}135e^{-x}(x^2+\frac{3}{2})$

75. Solve $(D^2+2D+4)y=e^x\sin 2x$ **Ans.** $y=e^{-x}(c_1\cos\sqrt{3}x+c_2\sin\sqrt{3}x)+\frac{e^x}{73}(3\sin 2x-8\cos 2x)$

76. Solve $(D^2+5D+6)y=e^{-2x}\sin 2x$ **Ans**. $y=c_1e^{-2x}+c_2e^{-3x}-\frac{1}{10}e^{-2x}(\cos 2x+2\sin 2x)$ **[June 16]**

77. Solve $(D^2+4)y=x\sin x$ **Ans**. $y=c_1\cos 2x+c_2\sin 2x+\frac{x}{3}\sin x-\frac{2}{9}\cos x$

78. Solve $(D^2-2D+1)y=x\sin x$ **Ans.** $y=(c_1+c_2x)e^x+\frac{(x+1)}{2}\cos x-\frac{1}{2}\sin x$ **[June 2006]**

79. Solve $(D^2-2D+1)y=xe^x\sin x$ **Ans.** $y=(c_1+c_2x)e^x-e^x(x\sin x+2\cos x)$ **[June 02, 08,Dec.08]**

80. Solve $(D^2-4D+4)y=8x^2e^{2x}\sin 2x$ **Ans.** $y=(c_1+c_2x)e^{2x}+e^{2x}[(3-2x^2)\sin 2x-4x\cos 2x]$ **[Dec03]**

SIMULTAANEOUS DIFFERENTIAL EQUATION

81. Solve $\frac{dy}{dt}-2x=e^{-t}$, $\frac{dy}{dt}-2x=e^{-t}$ **[Dec. 2002]**

82. Solve $Dx+2y=e^t$, $Dy-2x=e^{-t}$, **Ans.** $y=\frac{2}{5}e^t-\frac{1}{5}e^{-t}+c_1\sin 2t-c_2\cos 2t$

83. Solve $Dx+y=\sin t$, $Dy+x=\cos t$, $\frac{d}{dt}=D$,given that ,x=2 ,y =0 ,at t=0 ,

Ans. $y=\sin t+e^{-t}-e^t$, $x=e^t+e^{-t}$ **[June 02, 07,08,09 ,March 10, Dec. 2011 ,june2012, Dec.13, June 17, 18]**

84. Solve $Dx+Dy+3x=\sin t$, $Dx+y-x=\cos t$, **[Nov. 2018]**

85. Solve $Dx-7x+y=0$, $Dy-2x-5y=0$ where $\frac{d}{dt}=D$

Ans. $x = e^{6t}(c_1\cos t + c_2\sin t)$, $y = e^{6t}[(c_1 - c_2)\cos t + (c_1 + c_2)\sin t]$ **[RGPV DEC. 05,10,12]**

86. Solve $Dx + \omega y = 0$, $Dy - \omega x = 0$, $\frac{d}{dt} = D$ **Ans.** $x = c_1\cos\omega t + c_2\sin\omega t$, $y = -c_1\sin\omega t - c_2\cos\omega t$ **[Jan.06]**

87. Solve $Dx + 5x + y = e^t$, $Dy - x + 3y = e^{2t}$, $\frac{d}{dt} = D$ **[RGPV DEC. 03,14, June 16(CBCS)]**

Ans. $x = (c_1 + c_2t)e^{-4t} + \frac{4}{25}e^t - \frac{1}{36}e^{2t}$ $y = (c_1 + c_2t + c_3t^2)e^{-4t} + \frac{1}{25}e^t + \frac{7}{36}e^{2t}$

88. Solve $Dx = 2x + 6y$, $Dy = x + y$, **Ans.** $x = c_1e^{-t} + c_2e^{4t}$, $y = \frac{1}{2}c_1e^{-t} - \frac{1}{3}c_2e^{4t}$ **[June 06 ,DEC. 2011]**

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION OR CAUCHY'S EQUATION.

The differential equation of the type $x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots\ldots.. + a_ny = Q$, *where* $a_1, a_2, \ldots\ldots a_n$ are constants and Q is either constant or some function of *x.*

Put $x=e^z$ or $z = \log x$, $x\frac{d}{dx} = \frac{d}{dz} = D$, $x^2\frac{d^2}{dx^2} = D(D-1)$ ....and solve by previous methods

89. Solve $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$, **Ans.** $y = c_1x^4 + c_2x^{-1} - \frac{x^2}{6} - \frac{\log x}{2} - \frac{3}{8}$ **[Dec.02,June 07, DEC 11,13, June 15]**

90. Solve $(x^2D^2 + 2xD - 20)y = (x+1)^2$, **Ans.** $y = c_1x^{-4} + c_2x^5 - \frac{x^2}{18} - \frac{1}{10}x - \frac{1}{20}$ **[RGPV DEC 2010]**

91. Solve $(x^2D^2 + 2xD - 20)y = x^2$, **Ans.** $y = c_1x^{-4} + c_2x^5 - \frac{x^2}{18}$ **[RGPV June16(CBCS)]**

92. Solve $(x^2D^2 + xD - 1)y = x^2e^x$, **Ans.** $y = c_1x + c_2x^{-1} + \frac{xe^x}{2} - \frac{1}{2}(x - 2 + \frac{1}{x})e^x$ **[RGPV DEC 2011]**

93. Solve $(x^2D^2 - xD + 1)y = \log x$, **[June 18]**

94. Solve $(x^2D^2 + 5xD + 4)y = x\log x$, Ans $y = (c_1 + c_2\log x)x^{-2} + \frac{x}{27}(3\log x - 2)$.**[June 06]**

95. Solve $(x^2D^2 + 2xD - 12)y = x^3\log x$ **Ans.** $y = c_1x^3 + c_2x^{-4} + \frac{x^3}{98}[7(\log x)^2 - 2\log x]$ **[June 08, March10, Dec.12]**

96. Solve $(x^3D^3 + 2x^2D^2 + 2)y = 10(x + \frac{1}{x})$

Ans: $y = c_1x^{-1} + x[c_2\cos(\log x) + c_3\sin(\log x)] + 5x + \frac{2}{x}\log x$ **[June 03,07,April 09]**

97. Solve $(x^3D^3 + 3x^2D^2 + xD + 1)y = x + \log x$

Ans: $y = c_1x^{-1} + \sqrt{x}[c_2\cos(\frac{\sqrt{3}}{2}\log x) + c_3\sin(\frac{\sqrt{3}}{2}\log x)] + \frac{x}{2} + \log x$ **[June 2014]**

98. Solve $(x^2D^2 + xD + 1)y = \log x\sin(\log x)$,

Ans. $y = c_1\cos(\log x) + c_2\sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{1}{4}\log x\sin(\log x)$ **[Dec.05,Jan 06, DEC 08]**

99. Solve $(x^3D^3 + 3x^2D^2 + xD + 8)y = 65\cos(\log x)$ **Ans** $y = c_1x^{-1} + x[c_2\cos(\log x) + c_3\sin(\log x)] + 5x + \frac{2}{x}\log x$ **[Dec03]**

USEFUL FORMULAE For Unit-4

FACTORIZATION OF THE SUM OR DIFFRENCE OF TWO ANGLES FORMULAE

$(i)\ 2\sin A\cos B = \sin(A+B)+\sin(A-B)$, $(ii)\ 2\cos A\sin B = \sin(A+B)-\sin(A-B)$
$(iii)\ 2\cos A\cos B = \cos(A+B)+\cos(A-B)$, $(iv)\ 2\sin A\sin B = \cos(A-B)-\cos(A+B)$

(*MULTIPLE ANGLE*) *FORMULAE*

$(i)\ \sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$, $(ii)\ \tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

$(iii)\ \cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

$(iv)\ \sin 3A = 3\sin A - 4\sin^3 A$, $(v)\ \cos 3A = 4\cos^3 A - 3\cos A$, $(vi)\ \tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

HALF ANGLE FORMULA

$(i)\ \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \dfrac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$, $(ii)\ \tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

$(iii)\ \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = \dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

$(iv)\ 1-\cos A = 2\sin^2(A/2)$, $1+\cos A = 2\cos^2(A/2)$

HYPERBOLIC FUNCTIONS

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x}-1}{e^{2x}+1} = \dfrac{1-e^{-2x}}{1+e^{-2x}}$

$\text{sech}\, x = \dfrac{1}{\cosh x}$, $\text{cosech}\, x = \dfrac{1}{\sinh x}$, $\tanh x = \dfrac{1}{\coth x} = \dfrac{\sinh x}{\cosh x}$

$\cosh(-x) = \cosh x$ $\tanh(-x) = -\tanh x$

Log forms of hyperbolic functions :

$\cosh^{-1} x = \ln\left\{x+\sqrt{x^2-1}\right\}$, $x \ge 1$	$\sinh^{-1} x = \ln\left\{x+\sqrt{x^2+1}\right\}$, all x	$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), -1 < x < 1$

Properties of Hyperbolic Functions:

$\cosh^2 x - \sinh^2 x = 1$	$1-\tanh^2 A = \text{sech}^2 A$	$2\sinh^2 x + 1 = \cosh 2x$
$\sinh 2x = 2\cosh x \sinh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$	$2\cosh^2 x - 1 = \cosh 2x$
$\sinh(A+B) = \sinh A\cosh B + \cosh A\sinh B$	$\cosh(A+B) = \cosh A\cosh B + \sinh A\sinh B$	

Some Useful formulas: LIMIT OF SOME SPECIAL FUNCTIONS

(i) $\lim_{x\to\infty}\frac{1}{x}=0$ (ii) $\lim_{x\to-\infty}(1+\frac{1}{x})^{x}=e$ (iii) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(iv) $\lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{\tan^{-1}x}{x}=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{\tan^{-1}x}{x}$

(v) $\lim_{x\to\infty}\frac{e^{x}-1}{x}=1$ (vi) $\lim_{x\to\infty}\frac{a^{x}-1}{x}=In\ a\ ,a>0$ (v) $\lim_{x\to\infty}\frac{x^{n}-a^{n}}{x-a}=na^{n-1}$

INDETERMINATE FORMS $\frac{0}{0},\frac{\infty}{\infty},0\times\infty\ ,0^{0}\ ,\infty^{0}\ ,\infty-\infty\ ,1^{\infty}$ resolve indeterminate form before using the limit by using L-hospital rule or by solving the fractions.

DIFFERENTIAL AND INTEGRAL CALCULUS

First Principle: The *derivative* of the function *f(x)* is the function *f'(x)* defined by

$$f'(x)\equiv\frac{d}{dx}\left[f(x)\right]\equiv\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

S.No	Differentiation	Integration
1	$\frac{d}{dx}x^{n}=nx^{n-1}$	$\int x^{n}dx=\frac{x^{n+1}}{n+1}+c,\quad n\neq -1$
2	$\frac{d}{dx}e^{ax}=ae^{ax}$	$\int e^{ax}dx=\frac{e^{ax}}{a}$
3	$\frac{d}{dx}\log_{e}x=\frac{1}{x}$	$\int\frac{1}{x}dx=\log x$
4	$\frac{d}{dx}\log_{a}x=\frac{1}{x}\log_{a}e$	$\int a^{x}dx=\frac{a^{x}}{\log_{e}a}$
5	$\frac{d}{dx}\sin ax=a\cos ax$	$\int\sin ax\,dx=-\frac{\cos ax}{a}$
6	$\frac{d}{dx}\cos ax=-a\sin ax$	$\int\cos ax\,dx=\frac{\sin ax}{a}$
7	$\frac{d}{dx}\tan ax=a\sec^{2}ax$	$\int\tan ax\,dx=\frac{-\log\sec ax}{a}=\frac{\log\cos ax}{a}$ $\int\sec^{2}ax\,dx=\frac{\tan ax}{a}$
8	$\frac{d}{dx}\cot ax=-a\cos ec^{2}ax$	$\int\cot ax\,dx=\frac{-\log\cos ecax}{a}=\frac{\log\sin ax}{a}$ $\int\cos ec^{2}ax\,dx=\frac{-\cot ax}{a}$
9	$\frac{d}{dx}\sec ax\,dx=a\sec ax\tan ax$	$\int\sec ax\tan ax\,dx=\frac{\sec ax}{a}$ $\int\sec x\,dx=\log(\sec x+\tan x)=\log\ \tan(\frac{\pi}{4}+\frac{x}{2})$
10	$\frac{d}{dx}\cos ecax=-a\cos ec\,ax\cot ax$	$\int\cos ecax.\cot ax\ dx=\frac{-\cot ax}{a}$

		$\int \operatorname{cosec} x\,dx = \log(\operatorname{cosec} x - \cot x) = \log\tan\frac{x}{2}$
11	$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x$
12	$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = -\cos^{-1}x$
13	$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1}x$
14	$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = -\cot^{-1}x$
15	$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1}x$
16	$\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = -\operatorname{cosec}^{-1}x$
17	**MULTIPLICATION FORMULA** $\frac{d}{dx}f_1(x).f_2(x) = f_2(x).\frac{d}{dx}f_1(x) + f_1(x).\frac{d}{dx}f_2(x)$	**MULTIPLICATION FORMULA** $\int u.v\,dx = u\int v\,dx - \int\{\frac{d}{dx}u\,.\int vdx\}dx$ **Leibnitz′ successive integration by Parts** $= u\int vdx - u'\int\int v\,dx^2 + u''\int\int\int vdx^3 \ldots\ldots\ldots\ldots\underbrace{\int\int\int}_{n} vdx^n$
18	**DIVISION FORMULA(Quotient Rule)** $\frac{d}{dx}(\frac{f_1}{f_2}) = \frac{f_2.(\frac{d}{dx}f_1) - f_1.(\frac{d}{dx}f_2)}{(f_2)^2}$	
19	$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}}dx = \int x^{-1/2}dx = \frac{x^{1/2}}{1/2}$

Some Other Formulae for Integration

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\sin^{-1}\frac{x}{a}$	$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log(\frac{a+x}{a-x}) = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$, $-a < x < a$	
$\int \frac{1}{\sqrt{a^2+x^2}}dx = \log(x+\sqrt{a^2+x^2}) = \sinh^{-1}\left(\frac{x}{a}\right)$	$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log(x+\sqrt{x^2-a^2}) = \cosh^{-1}\left(\frac{x}{a}\right)$
$\int \sqrt{a^2-x^2}dx = \frac{1}{2}[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}]$	
$\int \sqrt{x^2+a^2}dx = \frac{1}{2}[x\sqrt{x^2+a^2} + a^2\log(x+\sqrt{x^2+a^2})]$	$\int \sqrt{x^2-a^2}dx = \frac{1}{2}[x\sqrt{x^2-a^2} + a^2\log(x-\sqrt{x^2+a^2})]$
$\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$	$\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

Differentiation and Integration of Hyperbolic Functions:

$f(x)$	$\sinh x$	$\cosh x$	$\tanh x$	$\operatorname{sech} x$	$\operatorname{cosech} x$	$\coth x$

$\frac{d}{dx} f(x)$	$\cosh x$	$\sinh x$	$\text{sec}^2\text{h}\, x$	$-\tanh x\, \text{sech}\, x$	$-\text{cosech}\, x \coth x$	$\text{cosech}^2\, x$
$\int f(x)dx$	$\cosh x$	$\sinh x$	$\log\cos hx$	$\tan^{-1}(\sin hx)$	$\log\tan x/2$	$\log\sin hx$

Definite Integral:

1. $\int_a^b f(x)\text{dx} = \int_a^b f(y)\text{dy} = \int_a^b f(t)\text{dt}$.
2. $\int_a^b f(x)\text{dx} = -\int_b^a f(x)\text{dx}$
3. $\int_a^a f(x)\text{dx} = \int_b^b f(x)\text{dx} = \int_a^b 0\,\text{dx} = 0$
4. Let $a \le c \le b$, then $\int_a^b f(x)\text{dx} = \int_a^c f(x)\text{dx} + \int_c^b f(x)\text{dx}$
5. (i) If $f(-x) = f(x)$ **(Even Function)** then $\int_{-a}^{a} f(x)\,\text{dx} = 2\int_0^a f(x)\,\text{dx}$

   (ii) If $f(-x) = -f(x)$ **(Odd Function)** then $\int_{-a}^{a} f(x)\,\text{dx} = 0$
6. If $f(x)$ is periodic function, with period T i.e. $f(x+T) = f(x)$

   (a) $\int_\alpha^\beta f(x)\text{dx} = \int_{\alpha+T}^{\beta+T} f(x)\text{dx}$ (b) $\int_0^\alpha f(x)\text{dx} = \int_T^{\alpha+T} f(x)\text{dx}$

Some Standard Results:

$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, $\int_0^\infty \frac{\cos x}{x}dx = \infty$, $\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}$,

$\int_{-\infty}^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}$, $\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, $\int_0^\infty e^{-ax}dx = \frac{1}{a}$,

$\int_0^\infty xe^{-x^2}dx = \frac{1}{2}$,